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13. ABSTRACT The purpose of this Trident Scholar project was to study a scalar field in sixdimensional Anti-de Sitter space by extending the Randall-Sundrum model. This model included a single scalar field and two compactified extra dimensions. One of these extra dimensions was defined by periodic boundary conditions. The other extra dimension was compactified and stabilized by a scalar field in the space. The shape of the six-dimensional space was defined by its metric, a mathematica structure that described how the length scale changes as a function of position in space and time. The metric was required to satisfy a differential equation known as the Einstein Field equation. By starting with some known facts about the structure of the metric, the Einstein equation was decomposed into a system of differential equations that were solved to find the final solution for the metric. In addition to the requirement of the Einstein Field equation, once the scalar field was added to the system, it needed to satisfy its own differential equation, the Klein-Gordon equation. Perturbation methods were used to simultaneously solve the Einstein Field equation and the Klein-Gordon equation to find the back reaction of the energy due to the scalar field on the six-dimensional Anti-de Sitter space metric. This process gave a new metric for the space that included the effect of the scalar field. The physical characteristics of the newly calculated space were explored to ensure that it satisfied the hierarchy problem as well as to determine how the laws of physics were affected by the warping of the space.

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6D Anti-de Sitter Space Solutions to Einstein's Field Equation with a Scalar Field

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Abstract

The purpose of this Trident Scholar project was to study a scalar field in six-dimensional Anti-de Sitter space by extending the Randall-Sundrum model. This model included a single scalar field and two compactified extra dimensions. One of these extra dimensions was defined by periodic boundary conditions. The other extra dimension was compactified and stabilized by a scalar field in the space.

The shape of the six-dimensional space was defined by its metric, a mathematical structure that described how the length scale changes as a function of position in space and time. The metric was required to satisfy a differential equation known as the Einstein Field equation. By starting with some known facts about the structure of the metric, the Einstein equation was decomposed into a system of differential equations that were solved to find the final solution for the metric.

In addition to the requirement of the Einstein Field equation, once the scalar field was added to the system, it needed to satisfy its own differential equation, the Klein-Gordon equation. Perturbation methods were used to simultaneously solve the Einstein Field equation and the Klein-Gordon equation to find the back reaction of the energy due to the scalar field on the six-dimensional Anti-de Sitter space metric. This process gave a new metric for the space that included the effect of the scalar field.

The physical characteristics of the newly calculated space were explored to ensure that it satisfied the hierarchy problem as well as to determine how the laws of physics were affected by the warping of the space.

Keywords:

Extra Dimensions

Randall-Sundrum Model

Hierarchy Problem

Anti-de Sitter

Contents

1 Motivation

Current theories in the physics community dictate that there are four fundamental forces in the universe. These four forces are the following:

1. Gravity - This is the force between two particles at a distance proportional to the masses of the two particles. In the theory of general relativity, it is the geometry of space that defines this force.

2. Electromagnetism - This is the force on particles in an electromagnetic field that acts at a distance. It is dependent on both the electrical charge of each particle and the speed at which they move relative to some fixed point.

3. Weak Nuclear Force - This force governs the interactions between fundamental particles in the nucleus of an atom such as protons and neutrons. It is the force responsible for the phenomenom of nuclear decay. This force only acts at distances similar to the diameter of a nucleus (about 10^{-15}m).

4. Strong Nuclear Force - This is the force between the nuclear particles, protons and neutrons, which effectively holds the nucleus of an atom together. Just like the Weak Nuclear Force, this force also only acts at distances similar to the diameter of a nucleus. [1]

Each of these forces have an energy scale associated with them which is used to compare the strength of the force relative to each of the other three forces. A method in particle physics for defining these energy scales involves manipulating the units of the scale describing each force into a unit of energy using the fundamental constants

$$\hbar = 1.055 \times 10^{-34}\ J \cdot s, \qquad c = 3.00 \times 10^{8}\ \frac{m}{s} \tag{1}$$

where $\hbar$ is Plank's constant and c is the speed of light. In the case of gravity, the scale of the force comes from Newton's Gravitation Law

$$F_g = \frac{Gm_1m_2}{r^2} \tag{2}$$

which relates the gravitational force, F_g in units of Newtons (N), to the masses, m_1 and m_2 with units kilograms (kg), and the distance, r with units of meters (m), between the two particles. In this equation, G, known as Newton's constant, acts as a proportionality constant and gives us the scale of the gravitational force. In the following method, [X] represents the units of X in basic units: kilogram, meter, and second.

$$G = 6.67 \times 10^{-11}\,\frac{N \cdot m^2}{kg^2},\ [G] = \frac{m^3}{kg \cdot s^2} \tag{3}$$

Since $[\hbar] = \frac{kg \cdot m^2}{s}$, Plank's constant can be used to eliminate the mass unit from the problem.

$$[G\hbar] = [G][\hbar] = \frac{m^3}{kg \cdot s^2}\frac{kg \cdot m^2}{s} = \frac{m^5}{s^3} \tag{4}$$

$$[\frac{G\hbar}{c^3}] = \frac{[G\hbar]}{[c]^3} = \frac{m^5}{s^3}\frac{s^3}{m^3} = m^2 \tag{5}$$

The term $\hbar c$ is used to convert distance units to energy units because its units are $J \cdot m$ where the Joule (J) is the unit of energy.

$$[\frac{\hbar c^5}{G}] = \frac{[\hbar c]^2}{[\frac{G\hbar}{c^3}]} = \frac{kg^2 \cdot \frac{m^6}{s^4}}{m^2} = kg^2 \cdot \frac{m^4}{s^4} = J^2 \rightarrow [\sqrt{\frac{\hbar c^5}{G}}] = J \tag{6}$$

Now there is a manipulation of G that gives units of energy. The next step is to substitute the numerical values for the constants and obtain the value for the gravitational energy scale in GeV.

$$\sqrt{\frac{\hbar c^5}{G}} = \sqrt{\frac{(1.055 \times 10^{-34}) \times (3.00 \times 10^8)^5}{6.67 \times 10^{-11}}} = 1.96 \times 10^9\ J \tag{7}$$

$$1.96 \times 10^9\ J \times \frac{1\ GeV}{1.602 \times 10^{-10}\ J} = 1.22 \times 10^{19}\ GeV \tag{8}$$

The result of this method is that the energy scale of the gravitational force is on the order of 10^{19}GeV. In order to find the energy scales for the other fundamental forces, notice the masses of the fundamental particles on which the forces act and convert these masses to energy values by using Einstein's famous principle

$$E = mc^2 \tag{9}$$

The particle associated with the Electromagnetic force is the electron which has mass (in energy units) of 0.511 MeV which is on the order of 10^{-3}GeV. The Strong Nuclear force has energy scale associated with the mass of protons and neutrons, which is on the order of 1 GeV. Finally, the Weak Nuclear force has energy scale associated with the mass of quarks, also on the order of 1 GeV.

While the Nuclear forces and the Electromagnetic force act with energy scales that are only slightly different, the Gravitational force is associated with an energy scale many orders of magnitude greater than the other three fundamental forces. This difference in energy scales accounts for the observation that, for example, the gravitational force between two electrons is many orders of magnitude smaller than the electric force between the two charged electrons at any given distance. The large disparity between gravity and the other forces is known in particle physics as the Hierarchy Problem. Current theories exist which have ways of eliminating this problem of hierarchy using extra dimensions of space. The focus of this paper is to expand these theories by including more extra dimensions and a scalar field to the system. [2]

2 Background Information

2.1 Indicies and Einstein's Summation Notation

The first step in finding a solution space is to understand the index notation and the equations used in General Relativity. In classical physics, vectors are thought of as having three components for a system in a three-dimensional space, one for each dimension; however, in relativistic analysis, it is useful to include a component of time in vectors, with units of distance, as a fourth component. An index notation is used to describe these vectors with the time component as the zeroth index. For example, dx^μ with μ=0, 1, 2, 3 represents the vector

$$dx^\mu = (dx^0, dx^1, dx^2, dx^3) \tag{10}$$

where dx^0 is the time component of the vector, and $dx^0 = cdt$ because [ct]=m. These vectors can also be written with lowered indices, dx_μ by multiplying by the metric(described below) of the associated space

$$\sum_{\mu=0}^{3} g_{\mu\nu} dx^\mu = dx_\nu = (dx_0, dx_1, dx_2, dx_3) \tag{11}$$

The index in this notation can be replaced by any symbol while retaining the same meaning as a vector; however, greek indices, such as μ and ν, will range over all dimensions of the system, while roman indices, such as i and j, will range over the standard 3 dimensions of space and the time dimension only. So if the system of the problem is a 5-dimensional space, $\mu = 0, 1, 2, 3, 4$ but $i = 0, 1, 2, 3$. [3]

The next useful definition is the Einstein summation notation. An Einstein summation is the index notation version of the scalar dot product of two vectors. Let dx^μ and dx^ν be two four-component vectors, then the scalar dot product of these two vectors is defined as

$$dx^\mu \cdot dx^\nu = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu = \sum_{\mu=0}^{3} dx_\mu dx^\mu \tag{12}$$

where the second equality comes from equation (11). Since this type of summation appears frequently, the summation symbol is dropped, and it is understood that the repetition of an index with one raised and one lowered in the same term implies summation over all values of the index. So, in the Einstein summation notation [4]

$$dx_\mu dx^\mu = \sum_{\mu=0}^{3} dx_\mu dx^\mu = dx_0 dx^0 + dx_1 dx^1 + dx_2 dx^2 + dx_3 dx^3 \tag{13}$$

This notation not only reduces the visual complexity of the equations used in the problem, it also generalizes the equations to make them valid for any number of dimensions by simply changing the range of the indices. [3]

2.2 Definition of the Metric

Using the index and summation notations it is now possible to understand the idea of a metric. The notion of length, or the distance between two points in a space is found by integrating the line element from point A to point B.

$$L = \int_A^B ds \tag{14}$$

where L is the length and ds is the line element which is defined in flat three-dimensional space with cartesian coordinates as

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{15}$$

Minkowski space is a flat space with the addition of a time coordinate that is used in both special relativity and general relativity. When expanding the flat space line element to include the time coordinate in the way of Minkowski space, it is written as

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2 \tag{16}$$

This can then be rewritten in index notation as

$$ds^2 = -dx^0dx^0 + dx^1dx^1 + dx^2dx^2 + dx^3dx^3 \tag{17}$$

This sum can be written using Einstein summation notation by defining a term with two indices which gives the coefficient of each term in the sum depending on the indices of that term. In the case of equation (17), this coefficient term, η must take the following values:

$$\begin{aligned}
\eta_{00} &= -1 \\
\eta_{11} &= 1 \\
\eta_{22} &= 1 \\
\eta_{33} &= 1 \\
\eta_{ab} &= 0 \text{ for } a \neq b
\end{aligned} \tag{18}$$

Using these values, the line element can be written in summation notation as

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu \tag{19}$$

where $\eta_{\mu\nu}$ can be viewed as representing the elements in a matrix in the μ-th row and the ν-th column. Then the matrix representation of $\eta_{\mu\nu}$ is

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{20}$$

This specific example is for a flat space; however, the line element can be written in the form of equation (19) for a space of any shape by changing the coefficient terms out front. In general the line element of a space is given as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{21}$$

where $g_{\mu\nu}$ is known as the metric for that space. Equation (20) is the metric for Minkowski space and is known as the Minkowski metric. The general metric can also be written in the form of a matrix and can be a function of the position in space, $g_{\mu\nu}(x^\alpha)$. [5]

2.3 Embedded Surfaces

A metric which describes a curved space, or any space that is not flat, can be described as a surface embedded inside of a higher dimensional flat space. The surface of a sphere is a curved 2-dimensional space; however, this can be thought of as existing inside of a flat, 3-dimensional space. Using this method, the non-flat metric for the surface of the sphere can be found by imposing a contraint equation on the metric for the flat space.

The metric for flat, 3-dimensional space in cartesian coordinates is

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{22}$$

Now the equation for the surface of a sphere which is used as the constraint equation is

$$x^2 + y^2 + z^2 = c^2 \tag{23}$$

where c is a constant representing the radius of the sphere. This equation is now solved for one of the coordinates, in this case z, and the differential of that coordinate is found in terms of the other two.

$$
\begin{aligned}
z &= \sqrt{c^2 - x^2 - y^2} \\
dz &= \frac{-x}{\sqrt{c^2 - x^2 - y^2}}dx + \frac{-y}{\sqrt{c^2 - x^2 - y^2}}dy \\
dz^2 &= \frac{x^2}{c^2 - x^2 - y^2}dx^2 + \frac{2xy}{c^2 - x^2 - y^2}dxdy + \frac{y^2}{c^2 - x^2 - y^2}dy^2
\end{aligned} \tag{24}
$$

Substituting dz^2 into equation (22) gives the non-flat metric for the 2-dimensional surface of a sphere.

$$
ds^2 = \frac{c^2 - y^2}{c^2 - x^2 - y^2}dx^2 + \frac{2xy}{c^2 - x^2 - y^2}dxdy + \frac{c^2 - x^2}{c^2 - x^2 - y^2}dy^2 \tag{25}
$$

This can then be written in the form of equation (21) with

$$
g_{\mu\nu} = \begin{pmatrix} \frac{c^2 - y^2}{c^2 - x^2 - y^2} & \frac{xy}{c^2 - x^2 - y^2} \\ \frac{xy}{c^2 - x^2 - y^2} & \frac{c^2 - x^2}{c^2 - x^2 - y^2} \end{pmatrix} \tag{26}
$$

This simple example can be generalized to any n-dimensional surface embedded in a sufficiently higher dimensional space.

2.4 Einstein's Field Equation

The metric of a space is dependent on the content of the space. A flat space like the Minkowski metric will exist in a vacuum; however, if the content of the space is not empty, the metric is found by solving the fundamental equation of general relativity, the Einstein Field Equation [5] given in index notation as

$$
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \lambda g_{\mu\nu} = -8\pi G T_{\mu\nu} \tag{27}
$$

In this equation, $g_{\mu\nu}$ is the metric of the space, G is Newton's gravitational constant from equation (3), $T_{\mu\nu}$ is the energy-momentum tensor, which contains the information about the energy content of the space, and λ is the cosmological constant. $R_{\mu\nu}$ and R are known as the Ricci curvature tensor and the Ricci curvature scalar, respectively. The Ricci curvature tensor and scalar are found with the procedure described below.

First define the affine connection [5],

$$\Gamma^{\sigma}_{\lambda\mu} = \frac{1}{2} g^{\nu\sigma} \left(\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} \right) \tag{28}$$

where the g with raised indices represents the inverse of the metric with lower indices. Equation (28) shows that the affine connection is simply a function of the metric and its first derivatives. The affine connection is then used to define the Reimann curvature tensor [6]

$$R^{\lambda}_{\mu\nu\kappa} = \frac{\partial \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\kappa}} - \frac{\partial \Gamma^{\lambda}_{\mu\kappa}}{\partial x^{\nu}} + \Gamma^{\eta}_{\mu\nu}\Gamma^{\lambda}_{\kappa\eta} - \Gamma^{\eta}_{\mu\kappa}\Gamma^{\lambda}_{\mu\eta} \tag{29}$$

From equations (28) and (29), we can see that the curvature tensor is constructed from only the metric tensor and first and second derivatives of the metric tensor. Now, by setting the indices λ and ν to be the same in equation (29) and applying the Einstein summation notation, we obtain the Ricci curvature tensor [5], $R_{\mu\nu}$, that appears in the Einstein Field Equation (27)

$$R_{\mu\kappa} = R^{\lambda}_{\mu\lambda\kappa} \tag{30}$$

From this definition for the Ricci tensor, the curvature scalar [5], R, is found by multiplying by the inverse of the metric

$$R = g^{\mu\kappa} R_{\mu\kappa} \tag{31}$$

3 5-Dimensional AdS and the Randall-Sundrum Model

Some theories that solve the hierarchy problem employ the use of extra spatial dimensions. One of these theories, called the Randall-Sundrum model, uses a 5-dimensional structure known as Anti-de Sitter space to construct a space which has the neccessary properties to solve the hierarchy problem.

3.1 Extra Dimensions

In extra-dimensional systems, the indices of the vectors and other tensors such as those used in defining the Einstein Field Equation range over all dimensions. The zero index term represents the time component, indices 1, 2, and 3 represent the components of the observable three spatial dimensions, and higher numbered indices respresent the components of the extra dimensions.

Since these extra dimensions are not seen in everyday life, they must be very small. While the normal three spatial dimensions are thought to extend to infinity in all directions, extra dimensions must be compactified, or have a small, finite length. One type of compactified extra dimension has the condition that all points separated by integer multiples of a finite interval are identified. These are known as periodic extra dimensions. Let x^4 be the coordinate of the extra dimension, then this condition is represented in the following way

$$x^4 = x^4 + nL \qquad n \in \mathbf{Z} \tag{32}$$

where L is some interval of fixed length. Since L is arbitrary, it can be represented as

$$L = 2\pi R \tag{33}$$

where R is the radius of a circle, and then the interval, L, is the circumference of that circle.

The extra dimension can then be visualized as a circle of radius R, where R will be very small to make the dimension compact and unobservable in everyday life. An important condition that arises from this type of compactification is that all functions of position must be well defined in the extra dimension. For example, the metric tensor, which can be a function of position, must satisfy

$$g_{\mu\nu}(x^4) = g_{\mu\nu}(x^4 + n2\pi R) \tag{34}$$

Another method of compactifying an extra dimension is to simply assign boundaries to the dimension separated by a small, finite distance. The distance of separation must be small enough that the extra dimension is still not observable at large distances. Also, the fact that there are boundaries imply that there must be boundary conditions for any term that is a function of position. For example, as the position in the extra dimension approaches the boundary, the metric tensor will approach some constant tensor:

$$g_{\mu\nu}(a) = A_{\mu\nu} \tag{35}$$

where a is the value of x^4 at the boundary and $A_{\mu\nu}$ is a constant tensor.

In order to better comprehend the idea of extra dimensions, it is useful to consider lower dimensional examples. First, consider a long, straight wire and an ant crawling on the wire. If this situation is seen from a distance much larger than the radius of the wire away, the wire will appear as a 1-dimensional line, and the ant may only crawl along the line. However, if viewed from a distance of the same order as the radius, the circumference of the wire becomes apparent, and the ant can be observed moving along the long and straight wire as well as around the circumference of the wire. If the ant walked the full length of the circumference, it would be back in the same location that it began. This situation can be thought of as an example of a compactified periodic extra dimension.

An example of the second type of compactified extra dimension is given by imagining the planet Earth. A human being on the surface of the Earth may move along lines of lattitude or lines of longitude but is confined to the two-dimensional surface of the Earth that exists at some constant radius. So the surface of the Earth is the normal observable dimensions for humans confined to that two-dimensional surface, and the radial direction is the extra dimension. Now imagine that a human gets in an airplane and takes off from the surface. This person is now free to move in the radial direction; however, the plane is only capable of flying to a certain altitude because it requires atmosphere to generate lift on the wings. Therefore, the surface of the Earth and the top limit of the atmosphere act as boundaries for the extra dimension. While in this example the extra dimension is not small, the analogy holds for a compactified dimension in which the boundaries of the extra dimension are a small distance apart.

3.2 Anti-de Sitter Space

Now that the concept of extra dimensions has been defined it is possible to look at an example of a 5-dimensional space. An important example of such a space is known as Anti-de Sitter Space. This space is the 5-dimensional vacuum solution (i.e. $T_{\mu\nu} = 0$) to Einstein's Field equation with a negative cosmological constant, $\lambda < 0$. [7]

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \lambda g_{\mu\nu} = 0,\ \lambda < 0 \tag{36}$$

In order to start solving the Einstein equation for the Anti-de Sitter space metric, it is neccessary to have a preliminary guess as to the form of the solution. The guess used in finding this solution was

$$ds^2 = -A(x^4)\eta_{ij}dx^i dx^j - dx^4 dx^4 \tag{37}$$

where $A(x^4)$ is an arbitrary function of the extra dimension, x^4 [8]. This gives

$$
\begin{aligned}
ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\
g_{\mu\nu} &= \begin{pmatrix} A(x^4) & 0 & 0 & 0 & 0 \\ 0 & -A(x^4) & 0 & 0 & 0 \\ 0 & 0 & -A(x^4) & 0 & 0 \\ 0 & 0 & 0 & -A(x^4) & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}
\end{aligned} \tag{38}
$$

Now that the guess for the metric is defined, the next step in solving the Einstein equation is to unpack the index notation to find differential equations for the function A in terms of the independent variable x^4. This process involves using the definitions in section 2.2 and the guess for the metric to write the Ricci tensor and the curvature scalar in terms of the function A and substituting them into equation (27). Once this is done, each combination of indices produces a differential equation in $A(x^4)$ that must be satisfied as a system of equations. Since the metric has only diagonal terms, the Einstein equation has non-trivial equations on the diagonal only. Using the Mathematica code found in Appendix A, these equations are found to be

$$
\begin{aligned}
\mu = \nu = 0, 1, 2, 3 \quad &\rightarrow \quad \frac{3}{2} \frac{A''(x^4)}{A(x^4)} - \lambda = 0 \\
\mu = \nu = 4 \quad &\rightarrow \quad \frac{3}{2} \left(\frac{A'(x^4)}{A(x^4)} \right)^2 - \lambda = 0
\end{aligned} \tag{39}
$$

This system of equations can be solved using standard differential equation solution methods, and the solutions are found to be

$$
A(x^4) = e^{\sqrt{\frac{2\lambda}{3}} x^4} \quad \text{or} \quad A(x^4) = e^{-\sqrt{\frac{2\lambda}{3}} x^4} \tag{40}
$$

Since the first function goes to ∞ as x^4 increases, this cannot be a solution, and the second function is the solution to the equations. Therefore, the Anti-de Sitter space metric is given by [8]

$$g_{\mu\nu} = \begin{pmatrix} e^{-\sqrt{\frac{2\lambda}{3}}x^4} & 0 & 0 & 0 & 0 \\ 0 & -e^{-\sqrt{\frac{2\lambda}{3}}x^4} & 0 & 0 & 0 \\ 0 & 0 & -e^{-\sqrt{\frac{2\lambda}{3}}x^4} & 0 & 0 \\ 0 & 0 & 0 & -e^{-\sqrt{\frac{2\lambda}{3}}x^4} & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{41}$$

3.3 Randall-Sundrum Model

The reason Anti-de Sitter space is such an important example of a 5-dimensional space is because it is used in a model created by Randall and Sundrum to solve the hierarchy problem [2]. In this model, the coefficient function, $A(x^4)$, is called the warp factor because it changes the length scale of the space as x^4 is varied. Specifically, at x^4=0, the metric is that of the normal observed universe, but as x^4 increases, the length scale decreases exponentially. From section 1 it is seen that length scale is inversely proportional to energy scale by a factor of $\hbar c$, so as x^4 increases the energy scale increases exponentially.

The Randall-Sundrum model takes advantage of this warp factor, $A(x^4)$, in the Anti-de Sitter space metric and positions the compactifying boundary the appropriate distance such that the energy scale ratio from one boundary to the other is approximately 10^{19}, the same ratio of the energy scale of gravity to the other three forces. Since from equation (14) we see that length is the integral of the line element, the model sets this boundaries so that at $x^4 = 0$

$$\int_A^B \sqrt{e^{-\sqrt{\frac{2\lambda}{3}}0}}dx^i \approx 1 \tag{42}$$

and at the boundary, $x^4 = c$,

$$\int_A^B \sqrt{e^{\sqrt{\frac{2\lambda}{3}}c}}dx^i \approx 10^{-19} \tag{43}$$

so that the length scale ratio between the two boundaries is the desired 10^{-19} and the energy scale ratio is thus 10^{19}.

The model then states that the particles for gravity, gravitons, are more likely to be located at the higher energy boundary instead of the at the boundary of the normally observed universe like the other three fundamental forces. The structure of this model accounts for the inconsistency in gravity's energy scale; however, it relies on the theorist to place an arbitrary boundary at the right location along the extra dimension. [2]

4 Extension to 6 Dimensions

4.1 A 6-Dimensional Metric

Just as with 5-dimensional space, in order to do calculations in a 6-dimensions, it is first necessary to have a guess for what the metric solution to Einstein's equation will be. The most logical choice for a 6-dimensional metric would be some extension of the 5-dimensional guess for the metric. Since a periodic extra dimension can be thought of as a circle with a radius R, the addition to the metric will come from the formula for the arclength of a circle

$$ds = Rd\theta \rightarrow ds^2 = R^2 d\theta^2 \tag{44}$$

Now θ is the 5th spatial dimension, so θ=x^5. Then if the radius of the extra dimension is allowed to vary as a function of the position along the warped extra dimension, x^4 then the 6-dimensional guess for the metric solution is

$$ds^2 = -A(x^4)\eta_{ij}dx^i dx^j - dx^4 dx^4 - B(x^4)dx^5 dx^5 \quad \text{where} \quad B(x^4) = R^2(x^4) \tag{45}$$

This can then be written as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$g_{\mu\nu} = \begin{pmatrix} A(x^4) & 0 & 0 & 0 & 0 & 0 \\ 0 & -A(x^4) & 0 & 0 & 0 & 0 \\ 0 & 0 & -A(x^4) & 0 & 0 & 0 \\ 0 & 0 & 0 & -A(x^4) & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -B(x^4) \end{pmatrix} \tag{46}$$

4.2 Equations of Motion

Once again, now that the guess for the metric has been defined, the index notation of Einstein's equation needs to be unpacked into a system of differential equations in A and B. Using the definitions in section 2.2, the guess for the 6-dimensional metric, and the Mathematica code found in Appendix A, the following system of differential equations was found. Note that A and B are functions of x^4 only, and the argument of these functions has been dropped for the sake of simplicity.

$$\begin{aligned} A\left(\frac{3}{2}\frac{A''}{A} + \frac{1}{2}\frac{B''}{B} + \frac{3}{4}\frac{A'B'}{AB} - \frac{1}{4}\left(\frac{B'}{B}\right)^2 - \lambda\right) &= -8\pi G T_{00} \\ -\frac{3}{2}\left(\frac{A'}{A}\right) - \frac{A'B'}{AB} + \lambda &= -8\pi G T_{44} \\ B\left(-\frac{1}{2}\left(\frac{A'}{A}\right)^2 - 2\frac{A''}{A} + \lambda\right) &= -8\pi G T_{55} \end{aligned} \tag{47}$$

The vacuum equations, in which the right side of all three equations is 0, can be solved exactly to find a 6-dimensional form of Anti-de Sitter space. The solutions and metric for the vacuum condition are

$$A = e^{-\sqrt{\frac{2\lambda}{5}}x^4}, \; B = e^{-\sqrt{\frac{2\lambda}{5}}x^4}$$

$$g_{\mu\nu} = \begin{pmatrix} e^{-\sqrt{\frac{2\lambda}{5}}x^4} & 0 & 0 & 0 & 0 & 0 \\ 0 & -e^{-\sqrt{\frac{2\lambda}{5}}x^4} & 0 & 0 & 0 & 0 \\ 0 & 0 & -e^{-\sqrt{\frac{2\lambda}{5}}x^4} & 0 & 0 & 0 \\ 0 & 0 & 0 & -e^{-\sqrt{\frac{2\lambda}{5}}x^4} & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -e^{-\sqrt{\frac{2\lambda}{5}}x^4} \end{pmatrix} \tag{48}$$

In the Anti-de Sitter space metric, the radius of the 6th dimension, given by $\sqrt{B}$ does not go to 0 until x^4 goes to ∞. This means that the space extends to infinite size and is not compactified. In order to change the shape of the space so that the radius goes to 0 at a finite value of x^4, there must be some energy in the system that causes the shape of the space to change. This energy can be added in the form of a scalar field so that the energy-momentum tensor is non-zero. This will give a non-zero right hand side of the equations of motion and force the solution space to be a different metric. This scalar field also adds an additional constraint to the system by adding a fourth differential equation to the system of equations already found from the Einstein equation. This equation, known as the Klein-Gordon equation, is the equation of motion for the scalar field and is given by

$$\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\mu}\left(\sqrt{|g|}g^{\mu\nu}\frac{\partial}{\partial x^\nu}\phi\right) = -\frac{\partial V(\phi)}{\partial \phi} \tag{49}$$

where $|g|$ is the determinant of the matrix form of the metric, ϕ is the scalar field, and $V(\phi)$ is the scalar field potential. For the purposes of this paper, it can be assumed that the scalar field is only a function of the position along the warped extra dimension, x^4. Given a known potential, the Einstein equaion and the Klein-Gordon equation must be solved simultaneously to find non-vacuum solutions for A, B, and ϕ as functions of x^4. [9]

4.3 Energy-Momentum Tensor

In order to find a non-vacuum solution space for the metric, it is first necessary to define the energy-momentum tensor in terms of the scalar field. This tensor is the term that makes the right hand side of the Einstein equation non-zero and gives rise to additional curvature in the solution space. The energy momentum tensor is defined as [5]

$$T_{\mu\nu} = \frac{\partial\phi}{\partial x^\mu}\frac{\partial\phi}{\partial x^\nu} - g_{\mu\nu}\left(\frac{1}{2}g^{\rho\sigma}\frac{\partial\phi}{\partial x^\rho}\frac{\partial\phi}{\partial x^\sigma} - V(\phi)\right) \tag{50}$$

Using this definition, the values for the energy-momentum tensor in equation (43) are found to be

$$\begin{aligned} T_{00} &= A\left(V(\phi) + \frac{1}{2}\phi'^2\right) \\ T_{44} &= -V(\phi) + \frac{1}{2}\phi'^2 \\ T_{55} &= -B\left(V(\phi) + \frac{1}{2}\phi'^2\right) \end{aligned} \tag{51}$$

where derivatives of ϕ with respect to other dimensions vanish because ϕ is only a function of x^4. The general system of differential equations for finding a non-vacuum solution space metric is

$$\begin{aligned} \mu,\nu = 0,1,2,3 \quad &\rightarrow \quad \frac{3}{2}\frac{A''}{A} + \frac{1}{2}\frac{B''}{B} + \frac{3}{4}\frac{A'B'}{AB} - \frac{1}{4}\left(\frac{B'}{B}\right)^2 - \lambda = -2\kappa V(\phi) - \kappa\phi'^2 \\ \mu,\nu = 4 \quad &\rightarrow \quad \frac{3}{2}\left(\frac{A'}{A}\right)^2 + \frac{A'B'}{AB} - \lambda = 2\kappa V(\phi) - \kappa\phi'^2 \\ \mu,\nu = 5 \quad &\rightarrow \quad \frac{1}{2}\left(\frac{A'}{A}\right)^2 + 2\frac{A''}{A} - \lambda = -2\kappa V(\phi) - \kappa\phi'^2 \\ &\phi'' + \left(\frac{1}{2}\frac{B'}{B} + 2\frac{A'}{A}\right)\phi' = -\frac{\partial V(\phi)}{\partial\phi} \end{aligned} \tag{52}$$

where $\kappa = 4\pi$G.

4.4 Solutions to the Equations

Ideally, a general solution to this system of equations could be found for A, B, and ϕ as functions of x^4 and in terms of integrals of the scalar field potential, $V(\phi)$. This would give formulae for these functions given any potential function, and allow for the trying of different scalar field potentials to see the effect on the solutions. However, each of these equations is non-linear in nature, and there is no standard mathematical method for solving non-linear differential equations.

Similar methods to finding the vacuum solution were tried but were unsuccessful due to the coupled nature of the Einstein and Klein-Gordon equations through the scalar field. Another method for solving these types of equations in 5 dimensions, known as the superpotential method, writes the scalar field potential in terms of another function of ϕ. It does this in such a way that the equations are decoupled into an equation for A and an equation for ϕ both in terms of the superpotential function. This was very effective for solving the 5-dimensional equations; unfortunately, since this method evolved from supersymmetry theory, it would require 6-dimensional supersymmetry calculations to attempt to extend this method to decouple the 6-dimensional system of equations. [10] These types of calculations are beyond the scope of this project.

It became clear that it would be difficult to find an exact analytical solution to the 6-dimensional system of equations for a general scalar field potential. An approximation method of solving the system of differential equations would be used.

5 Perturbation Theory Solution

An approximation method that met with success used perturbation theory to find small order corrections to the vacuum solution to Einstein's equation in six dimensions. This method mimicked the methods of Goldberger and Wise to find a small scalar field to compactify the x^4 dimension. This scalar field was then used to find the change in the metric. [11]

5.1 A Different Form of the Equations

In order to more easily work with the metric to find the back reaction due to the small scalar field used in the Goldberger-Wise mechanism, it is helpful to rewrite the current system of differential equations into a different form using a simple substitution.

$$\begin{aligned} A &= e^{\sigma} \\ B &= e^{\tau} \end{aligned} \tag{53}$$

By defining A and B this way, the first and second derivatives of these functions are

$$\begin{aligned} A' &= \sigma' e^{\sigma} \\ B' &= \tau' e^{\tau} \\ A'' &= \left((\sigma')^2 + \sigma'' \right) e^{\sigma} \\ B'' &= \left((\tau')^2 + \tau'' \right) e^{\tau} \end{aligned} \tag{54}$$

Now the system of differential equations in A, B, and ϕ becomes a system of equations in σ, τ, and ϕ. Again, all functions are functions of x^4 only unless otherwise specified.

$$\frac{3}{2}\sigma'' + \frac{1}{2}\tau'' + \frac{3}{2}\sigma'^2 + \frac{1}{4}\tau'^2 + \frac{3}{4}\sigma'\tau' - \lambda = 2\kappa V(\phi) + \kappa\phi'^2 \tag{55}$$

$$\frac{3}{2}\sigma'^2 + \sigma'\tau' - \lambda = 2\kappa V(\phi) - \kappa\phi'^2 \tag{56}$$

$$2\sigma'' + \frac{5}{2}\sigma'^2 - \lambda = 2\kappa V(\phi) + \kappa\phi'^2 \tag{57}$$

$$\phi'' + \left(\frac{1}{2}\tau' + 2\sigma' \right) \phi' = \frac{\partial V(\phi)}{\partial \phi} \tag{58}$$

This functional substitution does not constrain the system in any way by eliminating possible solutions, and it is useful because it eliminates the fraction terms in the original equations. This set of equations is the set that will be used to find the perturbation solution.

5.2 A Redundant Equation

Before a solution to the above system of differential equations is found, it is necessary to show that a solution can exist. Since there are three unknown functions to be solved, σ, τ, and ϕ, but there are four functions, it is possible that the system is overconstrained and no solutions would exist. In order for a solution to exist, there must be a redundancy in the equations, or one equation must be able to be written from the other three. To see this relationship, first take the derivative of both sides of equation (56) with respect to x^4.

$$3\sigma'\sigma'' + \sigma''\tau' + \sigma'\tau'' = 2\kappa V'(\phi)\phi' - 2\kappa\phi'\phi'' \tag{59}$$

Now add each side of equation (59) to each side of equation (58) multiplied by $2\kappa\phi'$.

$$3\sigma'\sigma'' + \sigma''\tau' + \sigma'\tau'' = (\tau' + 4\sigma')\kappa\phi'^2 \tag{60}$$

Now subtract equation (55) from equation (56) and multiply both sides by $2\sigma'$ to get

$$\frac{1}{2}\sigma'^2\tau' - 3\sigma'\sigma'' - \frac{1}{2}\sigma'\tau'^2 - \sigma'\tau'' = -4\kappa\sigma'\phi'^2 \tag{61}$$

Adding equations (60) and (61) cancels terms on each side of the equations.

$$\sigma''\tau' + \frac{1}{2}\sigma'^2\tau' - \frac{1}{2}\sigma'\tau'^2 = \tau'\kappa\phi'^2 \tag{62}$$

To get rid of the fractions and a factor of τ', multiply both sides of this equation by $\frac{2}{\tau'}$.

$$2\sigma'' + \sigma'^2 - \sigma'\tau' = 2\kappa\phi'^2 \tag{63}$$

Finally, adding equation (56) to equation (63) yields

$$2\sigma'' + \frac{5}{2}\sigma'^2 - \lambda = 2\kappa V(\phi) + \kappa\phi'^2 \tag{64}$$

which is the same as equation (57). This relationship shows that equation (57) can be written as a combination of equations (55), (56), and (58). Therefore, there is at least one redundant equation in the system, and so it can not be over constrained. A solution can exist and can be found using only 3 of the equations such that it is consistant with the remaining equation.

5.3 Zeroth Order Scalar Field Solution

The Goldberger-Wise technique involves placing boundary conditions on the scalar field function at $x^4 = 0$ and $x^4 = c$ where c eventually becomes the length of the compactified extra dimension x^4. This method then uses the zeroth order, or Anti-de Sitter space solution for A and B substituted into the Klein-Gordon equation with the simplest choice for the scalar field potential, $V(\phi) = m^2\phi^2$, to solve for the scalar field, ϕ. Since in Anti-de Sitter space $\frac{A'}{A} = \frac{B'}{B} = \sqrt{\frac{2\lambda}{5}}x^4$, the Klein-Gordon equation in equation (45) becomes

$$\phi'' - k\phi' - m^2\phi = 0 \tag{65}$$

with $k = \frac{1}{2}\sqrt{\frac{2\lambda}{5}} + 2\sqrt{\frac{2\lambda}{5}} = \frac{5}{2}\sqrt{\frac{2\lambda}{5}} = \sqrt{\frac{5\lambda}{2}}$. Using standard differential equations techniques, the solution to for the scalar field is found to be

$$\phi(x^4) = e^{\frac{kx^4}{2}}\left(C_1 e^{\nu\frac{kx^4}{2}} + C_2 e^{-\nu\frac{kx^4}{2}}\right) \tag{66}$$

where C_1 and C_2 are constants of integration defined by the boundary conditions and $\nu = \sqrt{1+\frac{4m^2}{k^2}}$. Using the assumption that $\frac{m^2}{k^2} << 1$ so that $\nu = 1+\epsilon, \epsilon = \frac{2m^2}{k^2}$ and the boundary conditions $\phi(0) = \phi_0$ and $\phi(c) = \phi_c$, the constants of integration are found to be

$$\begin{aligned} C_1 &= \phi_c e^{-(1+\nu)\frac{k}{2}c} - \phi_0 e^{-\nu kc} \\ C_2 &= \phi_0\left(1+e^{-\nu kc}\right) - \phi_c e^{-(1+\nu)\frac{k}{2}c} \end{aligned} \tag{67}$$

Substituting these constants into equation (47) gives the explicit solution for ϕ

$$\begin{aligned} \phi(x^4) = \quad & \phi_c\left(e^{(1+\nu)\frac{k}{2}(x^4-c)} - e^{\frac{k}{2}(x^4-c-\nu(x^4+c))}\right) \\ & -\phi_0\left(e^{\frac{k}{2}(1+\nu(x^4-2c))} - e^{(1-\nu)\frac{k}{2}x^4} - e^{\frac{k}{2}(1-\nu(x^4+2c))}\right) \end{aligned} \tag{68}$$

This solution for ϕ is the zeroth-order solution and because of the assumptions used, it is a small valued scalar field. The small value of the scalar field means that it will provide a small order perturbation from the Anti-de Sitter metric. [11]

5.4 A Non-Vacuum Solution

The technique to find the perturbation solution involves writing the final forms of σ and τ as a combination of the zeroth order (AdS) solution plus some small order correction term.

$$\begin{aligned} \sigma &= \sigma_0 + \alpha\sigma_1 \\ \tau &= \tau_0 + \alpha\tau_1 \end{aligned} \tag{69}$$

where $\alpha << 1$ and σ_1 and τ_1 are the correction functions to be solved for. It is important to note that α is of the same order as ϕ_0^2 and ϕ_c^2 in order to perform the perturbation technique. These relations for σ and τ as well as the solution for ϕ found in section 5.1 are substituted into equations (56) and (57)

$$\begin{aligned}\frac{3}{2}\sigma_0'^2 + 3\alpha\sigma_0'\sigma_1' + \frac{3}{2}\alpha^2\sigma_1'^2 + \sigma_0'\tau_0' + \alpha\sigma_0'\tau_1' & \\ +\alpha\sigma_1'\tau_0' + \alpha^2\sigma_1'\tau_1' - \lambda &= 2\kappa m^2\phi^2 - \kappa\phi'^2 \\ 2\sigma_0'' + 2\alpha\sigma_1'' + \frac{5}{2}\sigma_0'^2 + 5\alpha\sigma_0'\sigma_1' + \frac{5}{2}\alpha^2\sigma_1'^2 - \lambda &= -2\kappa m^2\phi^2 - \kappa\phi'^2 \end{aligned} \tag{70}$$

Since $\alpha << 1$, α^2 is negligable and is be assumed to be zero. Now, grouping terms with σ_0 and τ_0 only, we get

$$\begin{aligned}\left(\frac{3}{2}\sigma_0'^2 + \sigma_0'\tau_0' - \lambda\right) + 3\alpha\sigma_0'\sigma_1' + \alpha\sigma_0'\tau_1' + \alpha\sigma_1'\tau_0' &= 2\kappa m^2\phi^2 - \kappa\phi'^2 \\ \left(2\sigma_0'' + \frac{5}{2}\sigma_0'^2 - \lambda\right) + 2\alpha\sigma_1'' + 5\alpha\sigma_0'\sigma_1' &= -2\kappa m^2\phi^2 - \kappa\phi'^2 \end{aligned} \tag{71}$$

The expressions inside the parentheses are the left hand side of these equations for the vacuum solution, so the expressions inside the parentheses are equal to 0. Taking these terms to zero and substituting the known values for σ_0 and τ_0 into the remainder of the equation, they become

$$\begin{aligned}3\alpha\sqrt{\frac{2\lambda}{5}}\sigma_1' + \alpha\sqrt{\frac{2\lambda}{5}}\tau_1' + \alpha\sigma_1'\sqrt{\frac{2\lambda}{5}} &= 2\kappa m^2\phi^2 - \kappa\phi'^2 \\ 2\alpha\sigma_1'' + 5\alpha\sqrt{\frac{2\lambda}{5}}\sigma_1' &= -2\kappa m^2\phi^2 - \kappa\phi'^2 \end{aligned} \tag{72}$$

Now there is a linear system of inhomogeneous equations for σ_1 and τ_1 that are easily solved by standard differential equation solving techniques. The solutions are found to be

$$\begin{aligned}\sigma_1 = & -\frac{\kappa}{2}\Bigg(\left(\phi_c e^{-(1+\nu)\frac{k}{2}c} - \phi_0 e^{-\nu kc}\right)^2 \frac{\nu k^2 + 2m^2}{\nu(1+\nu)k^2} e^{(1+\nu)kx^4} \\ & +\left(\phi_c e^{-(1+\nu)\frac{k}{2}c} - \phi_0 e^{-\nu kc}\right)\left(\phi_0\left(1 + e^{-\nu kc}\right) - \phi_c e^{-(1+\nu)\frac{k}{2}c}\right) \\ & (4m^2 + (1-\nu)k^2)\left(\frac{x^4}{k} - \frac{1}{k^2}\right)e^{kx^4} \\ & +2\left(\phi_0\left(1 + e^{-\nu kc}\right) - \phi_c e^{-(1+\nu)\frac{k}{2}c}\right)^2 \frac{m^2}{\nu(1-\nu)k^2} e^{(1-\nu)kx^4}\Bigg) \end{aligned} \tag{73}$$

$$\begin{aligned}
\tau_1 &= \kappa\left(\phi_c e^{-(1+\nu)\frac{k}{2}c} - \phi_0 e^{-\nu kc}\right)^2 \left(\frac{\nu k^2 - 2m^2}{(1+\nu)k^2} + 2\frac{\nu k^2 + 2m^2}{\nu(1+\nu)k^2}\right) e^{(1+\nu)kx^4} \\
&+\kappa\left(\phi_c e^{-(1+\nu)\frac{k}{2}c} - \phi_0 e^{-\nu kc}\right)\left(\phi_0\left(1+e^{-\nu kc}\right) - \phi_c e^{-(1+\nu)\frac{k}{2}c}\right) \\
&\left(2(4m^2 + (1-\nu)k^2)\left(\frac{x^4}{k} - \frac{1}{k^2}\right) - \left(\frac{4m^2 - (1-\nu)k^2}{k^2}\right) e^{kx^4}\right. \\
&\left.+\kappa\left(\phi_0\left(1+e^{-\nu kc}\right) - \phi_c e^{-(1+\nu)\frac{k}{2}c}\right)^2 \left(\frac{4}{\nu} + 1\right) e^{(1-\nu)kx^4}\right) \qquad (74)
\end{aligned}$$

6 Analysis

It is necessary to determine the length of the x^4 dimension. This can be done using a method developed by Goldberger and Wise [11]. This procedure involves the calculus of variations which is reviewed below.

6.1 An Action Formulation

The action, S, is defined by an integral over time

$$S = \int_{t_1}^{t_2} L(q_i(t), \frac{dq_i(t)}{dt})dt \qquad (75)$$

where the Lagrangian, L, is a function of i generalized coordinates, $q_i(t)$, and their respective velocites, $\frac{dq_i(t)}{dt}$. Hamilton's principle states that any system will act so that the action is at a minimum or stationary value. It can be shown that the action is minimized at a solution to the Euler-Lagrange equations.

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \qquad (76)$$

These equations will, for the correct choice of q_i, return the well known form of Newton's force law, but it is also capable of giving equations of motions for much more complicated systems [13]. In particular, it can be generalized to relativistic systems and systems with fields.

Unlike a system of particle where the set of generalized coordinates is discrete, for a field, the set of generalized coordinates must be continuous. To include relativistic fields, the Langrangian in the action must be written as an integral of a Lagrangian density, $\mathcal{L}$, over all space.

$$S = \int_{t_1}^{t_2} \int \mathcal{L}(\phi, \partial_\mu \phi) d^5 x dt \tag{77}$$

In the case of the scalar field, the action is then given as

$$S = \frac{1}{2} \int_{t_1}^{t_2} \int \sqrt{g} \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - m^2 \phi^2 \right) d^5 x dt \tag{78}$$

The Euler-Lagrange equation from the action for a scalar field yields the Klein-Gordon equation.

6.2 Finding the Length of x^4

The scalar field action can now be used to find the the length of the x^4 dimension. The length of the x^4 dimension will be the value of c for which the action takes a minimum value by Hamilton's principle. The solution for ϕ found above is substituted into the action and integrated. The x^4 integral will have lower bound 0 and upper bound c, so the action will be a function of c, the length of the x^4 dimensions. Since the scalar field being substituted is a function only of x^4, all other spatial integrals will result in numerical factors that will divide out later. So it is now possible to think of the action as a function of c only. Basic calculus techniques find a minimum of a function by finding the zeros of the function's derivative. To find the value of c for which the action takes a minimum value, as required by Hamilton's principle, the derivative of the action with respect to c is set to zero and solved for a value of c. Since the action is essentially given as

$$\begin{aligned} S(c) &= \int_0^c \sqrt{g}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2\right) dx^4 \\ &= \int_0^c e^{\frac{4\sigma(x^4)+\tau(x^4)}{2}} \left(-(\phi'(x^4))^2 - m^2(\phi(x^4))^2\right) dx^4 \end{aligned} \tag{79}$$

the fundamental theorem of calculus shows that

$$\begin{aligned} \frac{d}{dc}S(c) &= \frac{d}{dc}\int_0^c e^{\frac{4\sigma(x^4+\tau(x^4)}{2}} \left(-(\phi'(x^4))^2 - m^2(\phi(x^4))^2\right) dx^4 \\ &= e^{\frac{4\sigma(c)+\tau(c)}{2}} \left(-(\phi'(c))^2 - m^2(\phi(c))^2\right) \end{aligned} \tag{80}$$

In this expression, σ and τ are the solutions to Einstein's equation with the perturbation corrections, and ϕ is the solution to the Klein-Gordon equation found earlier. By setting this derivative of the action equal to 0 and substituting those solutions into the equation, c, the length of the x^4 dimension can be determined in terms of the parameters of the system such as the mass of the scalar field and its boundary values.

6.3 The Hierarchy

The original purpose of the length scale variation of the Randall-Sundrum model was to explain the energy hierarchy between gravity and the other fundamental forces. The Randall-Sundrum model required that the x^4 dimension have an energy scale hierarchy of about nineteen orders of magnitude. Now that the space has an added extra dimension and further warping due to the scalar field, it is necessary to check that the new space still maintains the necessary length scale ratio to satisfy the hierarchy problem. In the metric, the warp factor is the square root of the coefficient function for the standard 4-space. So the length scale at any point along x^4 is given by the function $\sqrt{e^{\sigma(x^4)}}$. To test if this space has the required hierarchy, it is only necessary to compare the values of that function at each end of the space. For the following values of the system parameters: $m = .09k$, $\phi_0 = 1.5$, and $\phi_c = 1$, the length of x^4 is determined to be $c = 220$. These values also give length scale values at $x^4 = 0$, $\sqrt{e^{\sigma(0)}} = \mathcal{O}(1)$, and at $x^4 = c$, $\sqrt{e^{\sigma(c)}} = \mathcal{O}(10^{-19})$. This shows that the length scale ratio between the two ends of the x^4 dimension is approximately 10^{-19}, the nineteen orders or magnitude difference needed to satisfy the hierarchy problem.

6.4 Length Scale Effects

Another useful analysis of the changes made to the space by the scalar field perturbations is the effect on the relative size of the extra dimensions. In the following pictures, the two dimensional surface represents the two extra spatial dimensions. Every point on the surface contains a 4-dimensional space. For an observer at $x^4 = 0$ trying to measure the radius of the x^5 dimension at other positions along x^4, that observer would see the radius term in the metric, $\sqrt{e^{\tau(x^4)}}$. This would give the observer at $x^4 = 0$ the impression that the radius of x^5 falls of exponentially. This is depicted in Fig. 1.

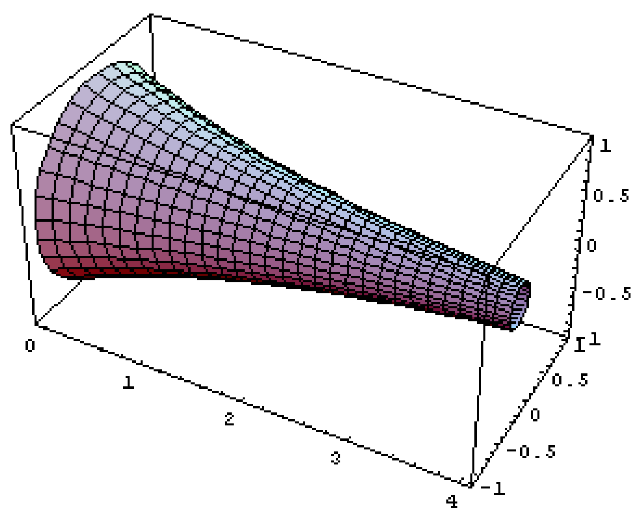

Figure 1: The apparent shape of the space for an observer stuck at $x^4 = 0$.

This point of view, however, does not effectively visualize how the shape of the space changes when the scalar field is added, since for perturbation theory, changes in σ and τ are small. A better way to view the shape of the space is from an observer moving along the x^4 dimension. As the observer moves along that warped extra dimension, the measurement of the radius of the x^5 dimension has to be relative to the length scale at that point. To see this more intuitively, the metric can be rewritten by factoring out the warp factor from all but the x^4 term.

$$ds^2 = e^\sigma \eta_{\mu\nu} dx^\mu dx^\nu - (dx^4)^2 - e^\tau (dx^5)^2 = e^\sigma \left(\eta_{\mu\nu} dx^\mu dx^\nu - \frac{e^\tau}{e^\sigma} (dx^5)^2 \right) - (dx^4)^2 \tag{81}$$

From this form of the metric, it can be seen that the rescaled radius of the x^5 dimension is given by $\sqrt{\frac{e^\tau}{e^\sigma}}$. For the 6-D AdS solution, the shape of the space as measured by an observer moving along x^4 is shown in Fig. 2. This figure shows that the radius appears constant. It

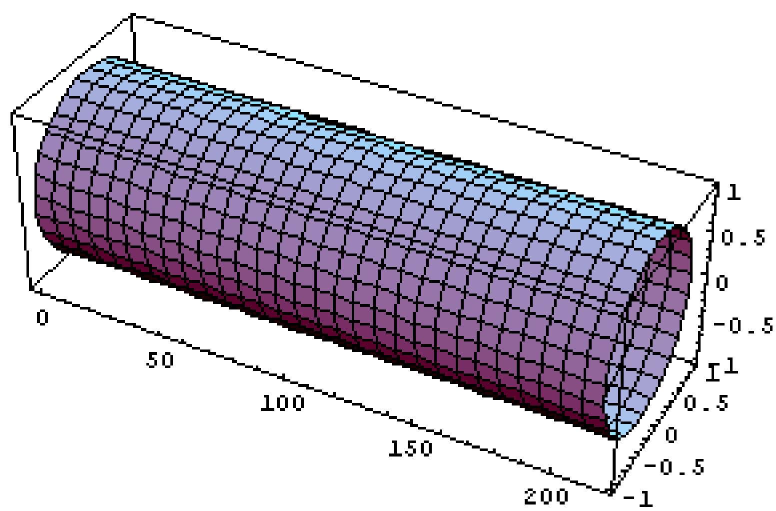

Figure 2: The shape of the AdS space for an observer moving on x^4

was determined above that, for the 6-D AdS space, $\sigma = \tau$, so the radius is a constant 1 at every point on x^4. When the perturbation solutions are plotted in the same way, the result is significantly different. Fig. 3 shows the shape of the perturbed space after including the scalar field. While the net effect on each function σ and τ is small under perturbation theory, it can have a large effect on the radius measured by an observer moving on x^4. Specifically, the radius appears to grow as the observer moves along the x^4 dimension.

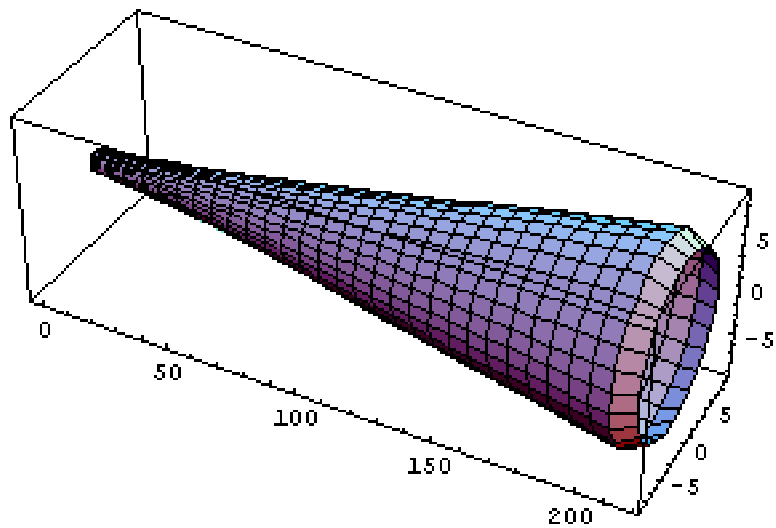

Figure 3: The shape of the perturbed space for an observer moving on x^4.

7 Effects on Physical Laws

An important aspect of understanding the characteristics of a space is to know how physical laws behave in the space. The best way to investigate how the warped space affects these physical laws is to examine a specific law and how it changes under the warping of the space. Some of the most well known and fully developed physical laws are Maxwell's equations of electromagnetism.

7.1 Classical Maxwell Equations

In classical physics, Maxwell's equations give a complete theory to describe the interactions of electromagnetic fields. These equations are typically viewed as four separate equations given from four separate physical laws. These fours laws are Gauss's law, Gauss's law for magnetism, Faraday's law, and Ampere's law and they are given by the following equations:

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho \tag{82}$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} \tag{83}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{84}$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial \vec{E}}{\partial t} \tag{85}$$

The first of these, Gauss's law, is typically used to derive the formula for the electric potential by using the fact that electric field $\vec{E}$ is the negative gradient of the potential, V_e.

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot (-\vec{\nabla} V_e) = \nabla^2 V_e = -4\pi\rho \tag{86}$$

For a stationary point particle, such as an electron, the solution to this differential equation gives the formula for the electric potential.

$$V_e(\vec{x}) = \frac{q}{4\pi\epsilon_0 \left|\vec{x} - \vec{x_0}\right|} \tag{87}$$

where $\vec{x_0}$ is the position of the particle. This formula shows that in classical physics, the electric potential attenuates as the inverse of the distance from the particle. This is useful for the classical, three-dimensional case; however, to derive formulas such as that for the electric potential in higher dimensional spaces or warped spaces, it is necessary to generalize these equations into index notation form. By defining an indexed vector known as the vector potential such that $A_\mu = (V_e, \vec{A})$ and the current density, $j^\nu = (4\pi\rho, \vec{j})$, Maxwell's equations can be rewritten as the following general equations:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{88}$$

$$\partial_\mu F^{\mu\nu} = 4\pi j^\nu \tag{89}$$

where $F_{\mu\nu}$ is called the Electromagetic field tensor. By examining these equations for the stationary point particle case above, it is determined that $A_\mu = (V_e, 0)$ and $j^\nu = (4\pi\rho, 0)$, and the equations reduce to the form of Gauss's law given above.

7.2 Generalized Maxwell Equations

Using this indexed form for the Maxwell equations, it can be extended to the six-dimensional model with warped spaces to give the formula for the electric potential in that space. However, since the space is warped and Maxwell's equations need to maintain the same form, all partial derivatives must be replaced with what is known as a covariant derivative. This derivative, designated by a capital D_μ, is the normal partial derivative plus an additional affine connection term for every order of the tensor. For example, the E-M tensor is of order two, so it's the above equation becomes

$$D_\mu F^{\mu\nu} = \partial_\mu F^{\mu\nu} + \Gamma^\mu_{\mu\sigma} F^{\sigma\nu} + \Gamma^\nu_{\mu\rho} F^{\mu\rho} \tag{90}$$

By breaking this down using the definitions in section 2.4, it can be shown that the covariant derivative of the E-M tensor can be given as

$$D_\mu F^{\mu\nu} = \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} F^{\mu\nu}\right) \tag{91}$$

Since the E-M field tensor is defined with lowered indices, the inverse tensor must be lowered in the above formula by multiplying by the inverse metric tensor.

$$\begin{aligned} F^{\mu\nu} &= g^{\mu\alpha} g^{\beta\nu} F_{\alpha\beta} = g^{\mu\alpha} g^{\beta\nu} \left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right) \\ D_\mu F^{\mu\nu} &= \frac{1}{\sqrt{g}} \partial_\mu \left(\sqrt{g} g^{\mu\alpha} g^{\beta\nu} \left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right)\right) = 4\pi j^\nu \end{aligned} \tag{92}$$

Since the purpose of this analysis is to determine the formula for the electric potential, the necessary equation is $\beta = 0$ and, because $g^{\beta\nu}$ is a diagonal tensor, ν must be 0 as well to have a non-trivial equation. These substitutions give

$$\frac{1}{\sqrt{g}}\partial_\mu\left(\sqrt{g}g^{\mu\alpha}g^{00}\left(\partial_\alpha A_0 - \partial_0 A_\alpha\right)\right) = 4\pi j^0 \tag{93}$$

This equation can be simplified by noticing that ∂_0 represents the time derivative, and A_α is constant in time, so that term is 0. Also, from definitions, $j^0 = c\rho$, $A_0 = V_e$, $g^{00} = e^{-\sigma}$, and $\sqrt{g} = e^{\frac{4\sigma+\tau}{2}}$. The new differential equation is then

$$e^{\frac{-(4\sigma+\tau)}{2}}\partial_\mu\left(e^{\frac{4\sigma+\tau}{2}}g^{\mu\alpha}e^{-\sigma}\partial_\alpha V_e\right) = 4\pi c\rho \tag{94}$$

The final step to getting the differential operator for V_e is to sum the repeated indices, α and μ, over all indices keeping in mind that $\alpha = 0$ gives a trivial equation since the vector potential is constant in time, and that the metric tensor is diagonal, so only terms with $\alpha = \mu$ give a non-trivial equation. The final differential equation then becomes

$$\left(e^{-2\sigma}\nabla^2_{3D} + e^{-\sigma}\frac{\partial^2}{(\partial x^4)^2} + e^{-\sigma}\left(\frac{2\sigma' + \tau'}{2}\right)\frac{\partial}{\partial x^4} + e^{-(\sigma+\tau)}\frac{\partial^2}{(\partial x^5)^2}\right)V_e = -4\pi c\rho \tag{95}$$

Now that the differential equation for the electric potential has been found for the warped space, it must be solved using the theory of Green's functions.

7.3 Finding a Green's Function

A standard method for solving the above type of second-order differential equations is by what is known as a Green's function. A Green's function is a function with the following property. Let D be a differential operator acting on some function $f(x)$ in a differential equation

$$Df(x) = h(x) \tag{96}$$

where $h(x)$ is a function of x that is the right side of the equation. Then the Green's function, $g(x, x')$ is defined so that

$$Dg(x, x') = \delta(x - x') \tag{97}$$

where $\delta(x - x')$ is a Dirac delta function. Once the Green's function is found from this new differential equation, then the solution to the original equation is given by

$$f(x) = \int g(x, x')h(x')dx' \tag{98}$$

This method could be used to find the solution to the above differential equation, but one of the major issues of this process is finding the Green's function for a complicated differential operator like the one above. One way to solve this type of problem is to decompose the Green's function and the delta function into linear combinations of the the eigenfunctions of D. Eigenfunctions are functions that, when operated on by the differential operator, return the same function multiplied by some scalar factor called the eigenvalue. So if

$$Df_n(x) = \lambda_n f_n(x) \tag{99}$$

then the Green's function and the delta function can be written as

$$\begin{aligned} G(x, x') &= \sum_{n=0}^{\infty} a_n(x')f_n(x) \\ \delta(x - x') &= \sum_{n=0}^{\infty} b_n(x')f_n(x) \end{aligned} \tag{100}$$

Now that the functions have been decomposed, the next step is to find the coefficients of the summations. This is done by using inner products and the fact that the eigenfunctions form an orthonormal basis. This means then that

$$\begin{aligned}\int f_m^*(x)\delta(x-x')dx &= \sum_{n=0}^{\infty} b_n(x') \int f_m^*(x) f_n(x) dx \\ f_m^*(x') &= \sum_{n=0}^{\infty} b_n(x')\delta_{mn} = b_m(x') \end{aligned} \tag{101}$$

where δ_{mn} is the Kroneker delta. Substituting this relation back into the definition for the delta function, the delta function is now written as

$$\delta(x-x') = \sum_{n=0}^{\infty} f_n^*(x') f_n(x) \tag{102}$$

Next, to find the coefficients of the Green's function decomposition, it is necessary to substitute the decompositions into the differential equation.

$$D\left(\sum_{n=0}^{\infty} a_n(x') f_n(x)\right) = \sum_{n=0}^{\infty} f_n^*(x') f_n(x)$$

Since the differential operator only acts on x and not x', the $a_n(x')$ coefficients are constant with respect to the operator. Therefore, the operator acts on the eigenfunctions and returns the same function back with the respective eigenvalue. Therefore the resulting equality becomes

$$\sum_{n=0}^{\infty} a_n(x') D f_n(x) = \sum_{n=0}^{\infty} a_n(x')\lambda_n f_n(x) = \sum_{n=0}^{\infty} f_n^*(x') f_n(x) \tag{103}$$

Once again using the fact that the eigenfunctions are orthogonal, the coefficients of each eigenfunction term must be equal, so

$$a_n(x') = \frac{f_n^*(x')}{\lambda_n} \tag{104}$$

Now the Green's function can be written explicitly as a summation of eigenfunctions and eigenvalues by

$$G(x,x') = \sum_{n=0}^{\infty} \frac{f_n^*(x')f_n(x)}{\lambda_n} \tag{105}$$

Finally, the solution to the initial differential equation can be found by getting the eigenfunctions of the differential operator and using Green's function theory to find the solution for $f(x)$.

7.4 Electric Potential in Warped Space

Now that the theory of Green's functions has been developed, it can be used to solve the"Gauss's Law" in the extra-dimensional warped space and find the formula for the electric potential in this space. Since the solution of the equation can be found by knowing the Green's function and the Green's function can be written as a sum of the eigenfunctions and eigenvalues, all that is left to do is solve for the eigenfunctions of the differential operator. For this specific example, the charge density will be a single point charge. This will give $\rho(x) = q\delta^5(x)$ where q is the magnitude of the charge and x' is the location of the point charge. The differential equation can then be rescaled by multiplying both sides by $e^{2\sigma}$ to give

$$\left(\nabla^2_{3D} + e^{\sigma}\frac{\partial^2}{(\partial x^4)^2} + e^{\sigma}\left(\frac{2\sigma' + \tau'}{2}\right)\frac{\partial}{\partial x^4} + e^{\sigma-\tau)}\frac{\partial^2}{(\partial x^5)^2}\right)V_e(x^\mu) = -4\pi q e^{2\sigma}\delta^5(x) \tag{106}$$

Since σ and τ are complicated functions in the perturbed space, solving this equation would be very difficult. For the sake of simplicity, the equation will be solved for the 6-D AdS space. This will still show that a warped, extra-dimensional space has an effect on the laws of physics. In this case $\sigma = \tau = \sqrt{\frac{2\lambda}{5}}x^4 = \alpha x^4$ where α is a constant. The eigenfunctions of the simplified operator will the solutions to the following equation

$$\left(\nabla^2_{3D} + e^{\alpha x^4}\frac{\partial^2}{(\partial x^4)^2} + e^{\alpha x^4}\frac{3}{2}\alpha\frac{\partial}{\partial x^4} + \frac{\partial^2}{(\partial x^5)^2}\right)f_n(x^\mu) = \lambda_n f_n(x^\mu) \tag{107}$$

Next, perform separation of variables by letting $f_n(x^\mu)$ be a product of functions of single variables

$$f_n(x^\mu) = X_1(x^1)X_2(x^2)X_3(x^3)X_4(x^4)X_5(x^5) \tag{108}$$

The result of this substitution is the equation

$$\frac{X_1''(x^1)}{X_1(x^1)} + \frac{X_2''(x^2)}{X_2(x^2)} + \frac{X_3''(x^3)}{X_3(x^3)} + e^{\alpha}\frac{X_4''(x^4)}{X_4(x^4)} + e^{\alpha}\frac{3}{2}\alpha\frac{X_4'(x^4)}{X_4(x^4)} + \frac{X_5''(x^5)}{X_5(x^5)} = \lambda_n \tag{109}$$

By standard methods in separation of variables, the first three terms and the last term are each equal to a separate constant $-w_i^2$. Therefore, the solutions are

$$X_1(x^1) = A_1 e^{iw_1x^1} \tag{110}$$

$$X_2(x^2) = A_2 e^{iw_2x^2} \tag{111}$$

$$X_3(x^3) = A_3 e^{iw_3x^3} \tag{112}$$

$$X_5(x^5) = A_5 e^{iw_5x^5} \tag{113}$$

Also, since the x^5 dimension has periodic boundary conditions, the w_5 eigenvalue is quantized as $w_5 = \frac{2n\pi}{L}$ where L is the circumference of the x^5 dimension. The differential equation can now be reduced to a function of x^4 only.

$$\frac{X_4''(x^4)}{X_4(x^4)} + \frac{3}{2}\alpha\frac{X_4'(x^4)}{X_4(x^4)} = e^{-\alpha x^4}(\lambda_n + w_1^2 + w_2^2 + w_3^2 + w_5^2) \tag{114}$$

Now that four out of the five solutions have been found, another method is required for finding the solution for $X_4(x^4)$ because it is not possible to write the eigenvalues explicitly. Going back to the formula for the Green's function, and substituting the known solutions into the formula, the expressions can be written as

$$\begin{aligned} G(x,x') &= \sum Ae^{i(w_1(x^1-x^{1\prime})+w_2(x^2-x^{2\prime})+w_3(x^3-x^{3\prime})+w_5(x^5-x^{5\prime}))}\frac{X_4^*(x^{4\prime})X_4(x^4)}{\lambda_n} \\ \delta(x-x') &= \sum Ae^{i(w_1(x^1-x^{1\prime})+w_2(x^2-x^{2\prime})+w_3(x^3-x^{3\prime})+w_5(x^5-x^{5\prime}))}\delta(x^4-x^{4\prime}) \end{aligned} \tag{115}$$

Since solutions for the X_4 functions are being looked for, these expressions can be simplified by grouping the functions into the two expressions

$$M(x^i,x^{i\prime},x^5,x^{5\prime}) = Ae^{i(w_1(x^1-x^{1\prime})+w_2(x^2-x^{2\prime})+w_3(x^3-x^{3\prime})+w_5(x^5-x^{5\prime}))} \tag{116}$$

$$N(x^4,x^{4\prime}) = \frac{X_4^*(x^{4\prime})X_4(x^4)}{\lambda_n} \tag{117}$$

An equation for $N(x^4,x^{4\prime})$ is found by putting going back to the Green's function equation

$$\begin{aligned} DG(x,x') &= \delta(x-x') \\ \left(\nabla_{3D}^2 + e^{\alpha x^4}\frac{\partial^2}{(\partial x^4)^2} + e^{\alpha x^4}\frac{3}{2}\alpha\frac{\partial}{\partial x^4} + \frac{\partial^2}{(\partial x^5)^2}\right)\left(\sum MN\right) &= \sum M\delta(x^4-x^{4\prime}) \\ \sum -(w_1^2+w_2^2+w_3^2+w_5^2)MN + e^{\alpha x^4}MN'' + e^{\alpha x^4}\frac{3}{2}\alpha MN' &= \sum M\delta(x^4-x^{4\prime}) \end{aligned}$$

By dividing out the M in each term and defining $\beta^2 = (w_1^2+w_2^2+w_3^2+w_5^2)$ a differential equation in terms of only x^4 is left.

$$e^{\alpha x^4}N'' + e^{\alpha x^4}\frac{3}{2}\alpha N' - \beta^2 N = \delta(x^4-x^{4\prime}) \tag{118}$$

This differential equation has solutions

$$N(x^4,x^{4\prime}) = e^{-\frac{3}{4}\alpha x^4}\left(CI_{\frac{3}{2}}(\frac{2\beta}{\alpha}e^{-\frac{\alpha}{2}}) + DK_{\frac{3}{2}}(\frac{2\beta}{\alpha}e^{-\frac{\alpha}{2}})\right) \tag{119}$$

where $I_{\frac{3}{2}}(\frac{2\beta}{\alpha}e^{-\frac{\alpha}{2}})$ and $K_{\frac{3}{2}}(\frac{2\beta}{\alpha}e^{-\frac{\alpha}{2}})$ are modified Bessel functions of the first and second kind, respectively. For the sake of simplicity of notation, these functions will be written from now on as I_x and K_x where x is the position on x^4 where the function is evaluated. Becuase this is the homogeneous solution the differential equation, it is the solution on either side of the delta function, and must satisfy boundary conditions are the singularity. First, the solution must be defined on either side of the singularity

$$N^< = e^{-\frac{3}{4}\alpha x^4}(C^< I_{x^4} + D^< K_{x^4})$$
$$N^> = e^{-\frac{3}{4}\alpha x^4}(C^> I_{x^4} + D^> K_{x^4})$$

and have then apply the following boundary conditions to find the four coefficients.

$$\begin{aligned}
N^<(x^{4\prime}) = N^>(x^{4\prime}) \quad &\rightarrow \quad (C^< I_{x^{4\prime}} + D^< K_{x^{4\prime}}) = (C^> I_{x^{4\prime}} + D^> K_{x^{4\prime}}) \\
\frac{\partial N^>(x^{4\prime})}{\partial x^4} - \frac{\partial N^<(x^{4\prime})}{\partial x^4} = e^{-\alpha x^{4\prime}} \quad &\rightarrow \quad (C^> - C^<)(I'_{x^{4\prime}} - \frac{3}{2}\alpha I_{x^{4\prime}}) \\
&\qquad +(D^> - D^<)(K'_{x^{4\prime}} - \frac{3}{2}\alpha K_{x^{4\prime}}) = e^{-\frac{1}{4}\alpha x^4} \\
\frac{\partial N^<(0)}{\partial x^4} = 0 \quad &\rightarrow \quad C^<(I'_0 - \frac{3}{2}\alpha I_0) - D^<(K'_0 - \frac{3}{2}\alpha K_0) = 0 \\
\frac{\partial N^>(c)}{\partial x^4} = 0 \quad &\rightarrow \quad C^>(I'_c - \frac{3}{2}\alpha I_c) - D^>(K'_c - \frac{3}{2}\alpha K_c) = 0
\end{aligned}$$

To further simplify the notation, define $\widetilde{I_x} = I'_x - \frac{3}{4}\alpha I_x$ and $\widetilde{K_x} = K'_x - \frac{3}{4}\alpha K_x$. The solutions for the coefficients are then

$$C^< \;=\; \widetilde{K_0}(I_{x^{4\prime}}\widetilde{K_c} - \widetilde{I_c}K_{x^{4\prime}})\frac{e^{-\frac{1}{4}\alpha x^{4\prime}}}{\frac{\alpha}{2}(\widetilde{I_c}\widetilde{K_0} - \widetilde{I_0}\widetilde{K_c})} \tag{120}$$

$$D^< \;=\; -\widetilde{I_0}(I_{x^{4\prime}}\widetilde{K_c} - \widetilde{I_c}K_{x^{4\prime}})\frac{e^{-\frac{1}{4}\alpha x^{4\prime}}}{\frac{\alpha}{2}(\widetilde{I_c}\widetilde{K_0} - \widetilde{I_0}\widetilde{K_c})} \tag{121}$$

$$C^> \;=\; \widetilde{K_c}(I_{x^{4\prime}}\widetilde{K_0} - \widetilde{I_0}K_{x^{4\prime}})\frac{e^{-\frac{1}{4}\alpha x^{4\prime}}}{\frac{\alpha}{2}(\widetilde{I_c}\widetilde{K_0} - \widetilde{I_0}\widetilde{K_c})} \tag{122}$$

$$D^> \;=\; -\widetilde{I_c}(I_{x^{4\prime}}\widetilde{K_0} - \widetilde{I_0}K_{x^{4\prime}})\frac{e^{-\frac{1}{4}\alpha x^{4\prime}}}{\frac{\alpha}{2}(\widetilde{I_c}\widetilde{K_0} - \widetilde{I_0}\widetilde{K_c})} \tag{123}$$

Now that there is a complete solution for the function $N(x^4, x^{4\prime})$, the Green's function can be completed and used to write the solution for the electric potential, V_e. The solutions for the electric potential would be given as

$$V_e = \int d^5x' \sum_{w_1,w_2,w_3,w_5} e^{i(w_1(x^1-x^{1\prime})+w_2(x^2-x^{2\prime})+w_3(x^3-x^{3\prime})+w_5(x^5-x^{5\prime}))}e^{-\frac{3}{4}\alpha x^4} (C^{>}(x^{4\prime})I_{x^4} + D^{>}(x^{4\prime})K_{x^4})e^{2\alpha x^{4\prime}}(-4\pi q)\delta^5(x^{\mu\prime}) \tag{124}$$

where C and D are given above depending on which side of the source the formula is being evaluated. Also, since the sum in this formula is a sum over all eigenvalues and β^2, inside the argument of the Bessel functions, is defined in terms of eigenvalues. The introduction of Bessel functions shows that the solution is affected by the extra dimensions and the warping of the space [14].

8 Conclusions

We used the dimensionally independent tensor form of Einstein's equation in 6-dimensions to find a system of equations for the coefficient functions in the metric. The vacuum solution to these equations gave a 6-dimensional form of Anti-de Sitter space with a warped extra dimension, x^4 and a periodic extra dimension, x^5. We then added a scalar field to the system that was forced to satisfy the Klein-Gordon equation. Using the methods of Goldberger and Wise, we were able to find the zeroth order solution for the scalar field. This zeroth order solution was then used to calculate the energy momentum tensor for the system of Einstein's equations and perturbation methods were used to find the back reaction on the metric coefficient functions. This gave a non-vacuum solution for the metric in 6-dimensions.

Once we found a non-vacuum perturbation solution to the metric, we continued to follow the methods of Goldberger and Wise by substituting the small order scalar field, $\phi(x^4)$, back into the action and evaluating the integral to get a formula for the action as a function of c, the length of the newly compactified extra dimension, x^4. Since, according to Hamilton's

Principle, the space must behave so that the action is takes its minimum value, standard calculus techniques were used to find the value of c where the action is a minimum. This value for c is the finite length of the x^4 dimension. For the space to remain consistent with the Randall-Sundrum model, the length of the x^4 dimension must be sufficient for the warp factor to provide the necessary energy scale difference to satisfy the hierarchy problem. It was determined that the warped space had the appropriate energy hierarchy for parameter values $m = .09k$, $\phi_0 = 1.5$, $\phi_c = 1$, and $c = 220$. [11]

The next step in the analysis of this perturbation solution was to see how the shape of the space was affected by the inclusion of a scalar field. This was checked by looking at the solution for the function $B(x^4)$. Since $B(x^4)$ represents the square of the radius of the periodic extra dimension, the extra-dimensional space was plotted as a surface of rotation. However, since the perturbation changes to $B(x^4)$ were required to be small, the plots of $\frac{B(x^4)}{A(x^4)}$ as the square of the radius were more effective at revealing the change that the scalar field had on the shape of the space. From these plots, it was evident that the radius of the space, as measured by an observer moving along x^4, grew along x^4 in the perturbed space.

Finally, we showed that the laws of physics can be affected by the addition of extra dimensions or warping in the space. Using the example of the electric potential for a stationary point charge, the formula was calculated for an extra-dimensional warped space. When compared to the classical formula for the electric potential of a stationary point charge, it is clear that the warping of the space and the extra dimensions have an effect on this law of physics.

The next step of this project would be to examine the effect of other types of field content on the metric. In this project, only the effect of a single scalar was examined. The same methods could be used to include a vector field, a tensor field, or any other higher rank field in the energy momentum tensor and examine the effect on the metric. Some combination of these fields may affect the metric in such a way that the metric has a natural boundary that closes off at one end. This would require the radius of the x^5 dimension to go to 0 at some value of $x^4 < c$ while still maintaining the energy hierarchy needed to stay consistent with the Randall-Sundrum model.

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